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Quasicrystal Ising Chain and Automata Theory

Jean-Paul Allouche[1] **and Michel Mendes France**[1]

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An automatic sequence is generated by a finite machine (automaton). These sequences can be periodic or not: in the latter case however, they are not random, but rather "quasicrystalline." We consider an Ising chain with variable interaction in a uniform external field, at zero temperature, and prove that, if this interaction is automatic, then the induced magnetic field is also automatic.

KEY WORDS: Ising chain; finite automata; substitutions.

1. INTRODUCTION

The cyclic Ising chain with variable interaction $\varepsilon_q J$ ($\varepsilon_q = \pm 1$), and uniform external field H is described through the Hamiltonian

$$\mathbf{H}(\mu) = -J \sum_{q=0}^{N-1} \varepsilon_q \mu_q \mu_{q+1} - H \sum_{q=0}^{N-1} \mu_q$$

where N is the number of sites, $\mu_q = \pm 1$, and $\mu_0 = \mu_N$. The ε_q's are given with prescribed values ± 1. The solution of the model involves the study of the partition function at temperature T:

$$\begin{aligned} Z_N(T) &= \sum_{\mu} \exp[-\beta \mathbf{H}(\mu)] \\ &= \sum_{\mu} \exp\left(\beta J \sum_{q=0}^{N-1} \varepsilon_q \mu_q \mu_{q+1} + H\beta \sum_{q=0}^{N-1} \mu_q\right) \end{aligned}$$

where $\beta = 1/kT$ is the inverse of the temperature and where the outer summation is extended over the 2^N different configurations $\mu = (\mu_0, \mu_1, ..., \mu_{N-1})$ in $\{-1, +1\}^N$.

[1] U.A. 226, U.E.R. de Math. et d'Info., Bordeaux I, 351, Cours de la Libération, 33405 Talence Cedex (France).

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Define

$$z = \exp(\beta J) \qquad \text{and} \qquad \alpha = H/J$$

and consider the transfer matrices

$$M_q = \begin{pmatrix} z^{\alpha + \varepsilon_q} & z^{\alpha - \varepsilon_q} \\ z^{-\alpha - \varepsilon_q} & z^{-\alpha + \varepsilon_q} \end{pmatrix}$$

It is well known that

$$Z_N(T) = \mathrm{Tr}\left(\prod_{q=0}^{N-1} M_q\right)$$

and that the free energy of the infinite chain is given by the thermodynamic limit

$$\lim_{N \to \infty} \frac{1}{N} \log\left(\mathrm{Tr} \prod_{q=0}^{N-1} M_q\right)$$

Following the ideas of B. Derrida (Derrida,[3] Derrida, Vannimenus, and Pomeau,[4] Gardner, and Derrida[6], we shall restrict our study to the behavior of the chain at the limit $T=0$.

The previous authors were interested in a random distribution of the ε_q's. Here we shall consider a distribution of the ε_q's generated by finite automata or substitutions (to be defined in Section 3). These sequences, known as "automatic" sequences, may be periodic or not. In the latter case, they stand somewhere between periodicity and randomness, closer however to periodicity. The structure is said to be "quasicrystalline."

We were very much influenced by F. Axel, M. Kléman, and J. Peyrière, whom we met once a month in Paris during the academic year 1984–1985. During these sessions we discussed together problems devoted to one-dimensional physics and automata theory. Independently of these meetings, we had some very fruitful discussions with Derrida. Finally G. Rauzy sent us an independent proof of Theorem 2. We express our warmest thanks to our five friends.

2. THE LIMIT $T=0$

As T vanishes, z increases to infinity and hence in the product of the matrices M_q one should only keep track of the higher powers of z. Given an arbitrary vector V_0 in R^2, define the vector

$$V_n = \prod_{q=0}^{N-1} M_q V_0$$

In the limit $T=0$,

$$V_n \sim \begin{pmatrix} p_n z^{a_n} \\ q_n z^{b_n} \end{pmatrix}$$

where p_n, q_n, a_n, and b_n are nonzero constants (independent of z).

The difference $d_n = a_n - b_n$ plays the role of the magnetic field of the chain at site n for $T=0$. In Ref. (3), Derrida observes that the double sequence a_n, b_n satisfies the recurrence relation

$$a_{n+1} = \max(a_n + \varepsilon_n, b_n - \varepsilon_n) + \alpha$$

$$b_{n+1} = \max(a_n - \varepsilon_n, b_n + \varepsilon_n) - \alpha$$

Our purpose is to study the sequence (d_n) when (ε_n) is a given automatic sequence. We shall demonstrate the following result:

Theorem 1. Consider an infinite quasicrystal Ising chain in a uniform external field. Let d_n be the magnetic field induced on the nth site at $T=0$. The sequence (d_n) is automatic. In other terms the quasicrystalline structure of the Ising chain induces a quasicrystalline output.

Our result is not surprising. The proof however is not completely trivial. We devote the following sections to some definitions and to the proof of Theorem 1, which, as we shall see, is an easy consequence of a general result concerning automatic sequences (our Theorem 2). Theorem 2 is in itself interesting and we hope to find other applications in the near future, possibly in the general theory of automata and in number theory.

3. SUBSTITUTIONS AND AUTOMATIC SEQUENCES

Let A be an alphabet, i.e., a finite set. Elements of A are called letters and finite sequences of letters are called words. Let $q \geqslant 2$ be a given integer. Consider a map (q-substitution) S from A to A^q and suppose that there exists a letter a in A such that the word $S(a)$ begins with a. Applying S to $S(a)$ means to replace every letter a_i of $S(a)$ by $S(a_i)$. One thus obtains the word $S^2(a)$ of length q^2. One can then consider the words $S^3(a)$, $S^4(a)$,... and so on. After infinitely many iterations, we thus obtain an infinite sequence $u=(u_0, u_1,...)$ in A^N which is invariant by S: $s(u)=u$. The sequence u is a fixed point of the mapping S.

Suppose there exist a set X and a map p from A to X, called projection. The sequence $p(u)=(p(u_0), p(u_1),...)$ in X^N is said to be generated by the substitution S and the projection p. We shall also say that this sequence is q-automatic (or simply automatic) or quasicrystalline.

Periodic sequences and ultimately periodic sequences are automatic. For example, the substitution S defined by

$$S(a) = aba$$

$$S(b) = bab$$

followed by the identity map p generates the periodic sequence

$$u = abababababababa...$$

The celebrated (nonperiodic) Thue–Morse sequence is generated in the same fashion:

$$S(a) = ab$$

$$S(b) = ba$$

$$p(a) = a$$

$$p(b) = b$$

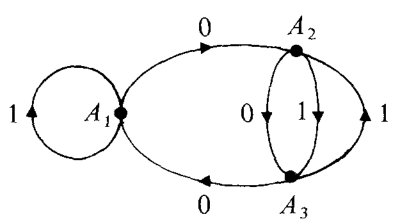
Its first terms are

$$abbabaabbaababbabaababbaabbabaabbaab...$$

Note that we have voluntarily limited our discussion to constant length substitutions, excluding by there Penrose-type substitutions.

4. AUTOMATA

Let $q \geqslant 2$ be a given integer. A q-automaton $\mathscr{A}$ is composed of a finite set of states $A_1, A_2,..., A_s$ (in the example below $q = 2$ and $s = 3$), A_1 is the initial state. q arrows $0, 1,..., q-1$ join each state to a state, and an output function T maps the set of states $\{A_1, A_2,..., A_s\}$ into some finite set X:



$$T(A_1) = T(A_2) = x$$

$$T(A_3) = y$$

The automaton acts on the sequence of integers 0, 1, 2,... as follows. Let n be an integer which we express in base q:

$$n = d_{k-1} d_{k-2} \cdots d_1 d_0, \quad d_i = 0, 1,..., q-1, \quad \text{and } d_{k-1} \neq 0 \quad \text{if } n \neq 0$$

Starting from the initial state A_1, we follow the instructions $d_{k-1}, d_{k-2},..., d_1, d_0$ in that order, sending the automaton from state A_1 to some final state A_j after k operations.

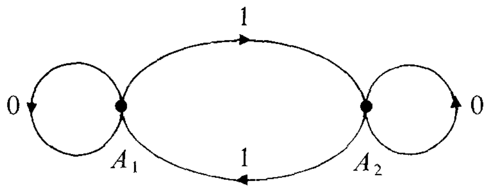
For instance, if n = nineteen = 10011, then on our example

$$A_1 \xrightarrow{1} A_1 \xrightarrow{0} A_2 \xrightarrow{0} A_3 \xrightarrow{1} A_2 \xrightarrow{1} A_3$$

We then read off the output $T(A_3) = y$. To every integer n corresponds an element x_n of X. We say that the infinite sequence $(x_0, x_1,...)$ in X^N is generated by the automaton $\mathscr{A}$. In our example the sequence begins with

$$xxxxyyxxxxxxxyyxxx...$$

The Thue–Morse sequence is generated by the following 2-automaton:



$T(A_1) = a$

$T(A_2) = b$

so that we now have two independent definitions of the same sequence. Actually the situation is quite general. It can be indeed shown that *a sequence is generated by q-automaton if and only if it is generated by a q-substitution and a projection* (see Cobham[2] or Christol *et al.*,[1] where the relationship with regular languages is also discussed).

We now describe another general result concerning automatic sequences and which will be crucial in the proof of Theorem 1.

5. MORE ABOUT AUTOMATIC SEQUENCES

Let X be an alphabet on which is defined an associative operation $*$.

Theorem 2. Let $x = (x(n))$ be a q-automatic sequence on the alphabet X. The sequence $y = (y(n))$ defined by

$$y(1) = x(0)$$
$$y(2) = x(1) * x(0)$$
$$\vdots$$
$$y(n) = x(n-1) * x(n-2) * \cdots * x(0)$$

is q-automatic.

Proof. We first add, if necessary, a neutral element e to X; we will then prove that the sequence $(y(n))_{n\geqslant 0}$ is q automatic, where for example $y(0)=e$. The proof proceeds in four steps:

(i) We first introduce the sequences $x_1, x_2, x_3,...$ obtained by grouping q (resp. $q^2, q^3,...$) terms of x together and calculating their $*$ product. We then define the sequences $y_j=(y_j(n))$ by

$$y_j(n)=x_j(n-1)*\cdots*x_j(0)$$

(ii) We then prove that the sequences x_j are "uniformly q-automatic" (i.e., obtained by projections from the same fixed point of a certain q-substitution).

(iii) We deduce that the set of all sequences y_j is actually finite.

(iv) We finally construct a finite set of sequences W, containing the y_j's and with the property: for every v in W, for every r in $[0, q-1]$, the sequence $n\to v(qn+r)$ belongs to W.

This will prove (see Ref. 5, p. 107) that all the sequences y_j, and hence y, are q-automatic.

(a) *The x_j and the y_j.* We define

$$\forall n\geqslant 0, \qquad x_0(n)=x(n)$$

$$\forall n\geqslant 0,\ \forall j\geqslant 0, \qquad x_{j+1}(n)=x_j(qn+q-1)*x_j(qn+q-2)*\cdots*x_j(qn)$$

$$\forall n\geqslant 1, \qquad y_j(n)=x_j(n-1)*\cdots*x_j(0)$$

$$\forall j, \qquad y_j(0)=e$$

(hence $y_0=y$)

Remark. For every $n\geqslant 0$, we have $y_j(qn)=y_{j+1}(n)$. The result is clear for $n=0$; for $n\geqslant 1$ we have

$$\begin{aligned} y_j(qn) &= x_j(qn-1)*\cdots*x_j(0)\\ &= (x_j(q(n-1)+q-1)*\cdots*x_j(q(n-1)))*\cdots*\\ &\quad (x_j(q-1)*\cdots*x_j(0))\\ &= x_{j+1}(n-1)*\cdots*x_{j+1}(0)\\ &= y_{j+1}(n) \end{aligned}$$

(b) *The sequences x_j are uniformly q-automatic.* More precisely, we prove the following proposition:

Proposition. There exist an alphabet B, a q-substitution $S = S_0 S_1 \cdots S_{q-1}$ on B, (where S_i is a map on B) and a sequence $(b(n))_n$ with terms in B such that

$$\forall j \geqslant 0 \quad \exists f_j\colon \quad B \to A \qquad \forall n \geqslant 0 \quad f_j(b(n)) = x_j(n)$$

The main theorem of Christol *et al.*[1] provides us with an alphabet B, a q-substitution $S = S_0 S_1 \cdots S_{q-1}$ on B, a sequence $(b(n))$ with terms in B and a map f from B to A, such that (i) the sequence b is a fixed point of S, i.e., $\forall r \in [0, q-1]$, $S_r(b(n)) = b(qn+r)$; (ii) for every $n \geqslant 0$, $f(b(n)) = x(n)$.

This proves our proposition for $j=0$, with $f_0 = f$. Let us prove by recurrence on j that this choice of B, S, b fits for every j:

To get f_{j+1} from f_j, it suffices to put

$$f_{j+1}(x) = f_j(S_{q-1}(x)) * \cdots * f_j(S_0(x))$$

hence

$$\begin{aligned} f_{j+1}(b(n)) &= f_j(S_{q-1}(b(n))) * \cdots * f_j(S_0(b(n))) \\ &= f_j(b(qn+q-1)) * \cdots * f_j(b(qn)) \\ &= x_j(qn+q-1) * \cdots * x_j(qn) \\ &= x_{j+1}(n) \end{aligned}$$

(c) *there is a finite number of sequences* y_j. There is a finite number of sequences x_j: the proposition above defines an injection from the set of the sequences x_j into the set of the maps from B to A $(x_j \to f_j)$; hence the finiteness of the y_j set.

(d) *the set* W. Define W to be set of all sequences $n \to (g(b(n)) * y_j(n))$, where g is an application from B to A and j an integer. This set is finite: g and y_j run through finite sets; moreover it contains y (take g constant equal to e and $j=0$).

Finally if $(v(n))$ is a sequence in W, then for every r in $[0, q-1]$ let us prove that the sequence $n \to v(qn+r)$ also lies in W: if

$$v(n) = g(b(n)) * y_j(n)$$

then

$$\begin{aligned} v(qn+r) &= g(b(qn+r)) * y_j(qn+r) \\ &= g(S_r(b(n))) * [x_j(qn+r-1) * \cdots * x_j(qn)] * y_j(qn) \end{aligned}$$

(the bracket is empty for $r=0$); hence

$$\begin{aligned} v(qn+r) &= g(S_r(b(n))) * [f_j(S_{r-1}(b(n))) * \cdots * f_j(S_0(b(n)))] * y_j(qn) \\ &= h(b(n)) * y_{j+1}(n) \end{aligned}$$

where h is defined by

$$h(x) = g(S_r(x)) * [f_j(S_{r-1}(x)) * \cdots * f_j(S_0(x))]$$

Our Theorem 2 is thus established. ▮

6. PROOF OF THEOREM 1

Recall that (a_n) and (b_n) satisfy the recursion formulas:

$$\begin{cases} a_{n+1} = \alpha + \max\{a_n + \varepsilon_n, b_n - \varepsilon_n\} \\ b_{n+1} = -\alpha + \max\{a_n - \varepsilon_n, b_n - \varepsilon_n\} \end{cases}$$

The magnetic field $d_n = a_n - b_n$ is then the solution of the recursion

$$\begin{cases} d_{n+1} = 2\alpha + \varepsilon_n \operatorname{sgn}(d_n) \min\{2, |d_n|\} \\ d_0 \quad = a_0 - b_0 \end{cases} \tag{1}$$

where sgn(.) is the sign function.

Clearly

$$\forall n \geqslant 1, \qquad 2\alpha - 2 \leqslant d_n \leqslant 2\alpha + 2$$

Moreover it is easy to prove that

$$\forall n \geqslant 0, \qquad d_n \in \{2n\alpha \pm d_0; n \in Z\} \cup \{2n\alpha \pm 2; n \in Z\}$$

so that d_n takes only finitely many values. Call Y the set of values of d_n. Define two maps on Y by

$$f_1(x) = 2\alpha + (\operatorname{sgn} x) \min\{2, |x|\}$$

$$f_{-1}(x) = 2\alpha - (\operatorname{sgn} x) \min\{2, |x|\}$$

Relation (1) can then be written

$$d_{n+1} = f_{\varepsilon_n}(d_n)$$

so that

$$d_{n+1} = f_{\varepsilon_n} \circ f_{\varepsilon_{n-1}} \circ \cdots \circ f_{\varepsilon_0}(d_0) \qquad \text{for every } n \geqslant 0$$

It is now obvious from Theorem 1 that the sequence (g_n) of maps defined by

$$g_0 = id$$
$$g_{n+1} = f_{\varepsilon_n} \circ f_{\varepsilon_{n-1}} \circ \cdots \circ f_{\varepsilon_0} \qquad \text{for every } n \geqslant 0$$

is automatic, provided (ε_n) is automatic. The sequence $d_{n+1} = g_{n+1}(d_0)$ is thus automatic.

7. AN EXPLICIT EXAMPLE

Suppose $\alpha = 1$, then

$$d_{n+1} = 2 + \varepsilon_n \operatorname{sgn}(d_n) \min\{2, |d_n|\}$$

Choose $d_0 = 2$. The above relation shows that d_n takes its values in $\{0, 2, 4\}$. It is then natural to define:

$$d'_n = -1 + (d_n/2) \in \{-1, 0, 1\}$$

whence

$$d'_{n+1} = \varepsilon_n(1 + \min\{0, d'_n\}) = f'_{\varepsilon_n}(d'_n)$$
$$d'_0 = 0$$

We now assume that the sequence (ε_n) is the Thue–Morse sequence defined on the symbols $+$ and $-$,

$$(\varepsilon_n) = + - - + - + + - - + + - + - - + - + + - + - \cdots$$

the sequence $d' = (d'_n)$ hence begins as follows:

$$(d'_n) = 0 + - 0 + - 0 + - 0 + + - 0 - 0 + - 0 + - 0 - 0 + + - 0 + - 0 \cdots$$

Actually the first two terms are, to some extent, inessential: had we choosen other initial conditions, the resulting sequences would coincide with (d'_n) from d'_2 on.

Instead of studying (d'_n) which, according to Theorem 2, seems to require a 108-state automaton, we rather consider the shifted sequence (d'_{n+2}) which happens to be much simpler. Needless to say (d'_n) is automatic if and only if (d'_{n+2}) is itself an automatic sequence.

Let us thus prove that the sequence (d'_{n+2}) is generated by the substitution defined on the alphabet $\{A, B, C, D, E\}$ by

$$S(A) = AB$$
$$S(B) = CA$$
$$S(C) = DE$$
$$S(D) = CE$$
$$S(E) = AD$$

followed by the projection P

$$P(A) = -$$
$$P(B) = P(D) = 0$$
$$P(C) = P(E) = +$$

Let (u_n) be the fixed point of S, hence

$$(u_n) = (A\ B\ C\ A\ D\ E\ A\ B\ C\ E\ A\ D\ A\ B\ C\ A\ D\ E\ A\ D \cdots)$$

and let (v_n) be the pointwise projection of (u_n), hence

$$\forall n \geqslant 0, \qquad v_n = P(u_n)$$

We claim that

$$\forall n \geqslant 0, \qquad v_n = d'_{n+2}$$

First of all, let S_0 and S_1 be the components of S, i.e.,

$$S_0(A) = A, \qquad S_1(A) = B$$
$$S_0(B) = C, \qquad S_1(B) = A$$
$$S_0(C) = D, \qquad S_1(C) = E$$
$$S_0(D) = C, \qquad S_1(D) = E$$
$$S_0(E) = A, \qquad S_1(E) = D$$

As (u_n) is the fixed point of S, we then have

$$\forall n \geqslant 0, \qquad u_{2n} = S_0(u_n)$$
$$u_{2n+1} = S_1(u_n)$$

At this point the patient reader can check the following (nonindependent) relations:

$$
\begin{aligned}
&S_0^4=S_0^2, && S_0^3S_1=S_0S_1\\
&S_1S_0^3=S_1S_0, && S_1S_0^2S_1=S_1S_0S_1\\
&S_0S_1S_0^2=S_0S_1S_0, && (S_0S_1)^2=S_0S_1^2\\
&S_1^2S_0^2=S_1^2S_0, && S_1^2S_0S_1=S_0^2S_1\\
&S_0^2S_1S_0=S_0^2S_1, && S_0^2S_1^2=S_1^2S_0\\
&(S_1S_0)^2=S_1S_0S_1, && S_1S_0S_1^2=S_1S_0\\
&S_0S_1^2S_0=S_0S_1^2, && S_0S_1^3=S_0S_1\\
&S_1^3S_0=S_1S_0, && S_1^4=S_1^2
\end{aligned}
$$

These relations, respectively, imply, for every $n\geqslant 0$

$$
\begin{aligned}
&u_{16n}=u_{4n}, && u_{16n+8}=u_{4n+2}\\
&u_{16n+1}=u_{4n+1}, && u_{16n+9}=u_{8n+5}\\
&u_{16n+2}=u_{8n+2}, && u_{16n+10}=u_{8n+6}\\
&u_{16n+3}=u_{8n+3}, && u_{16n+11}=u_{8n+4}\\
&u_{16n+4}=u_{8n+4}, && u_{16n+12}=u_{8n+3}\\
&u_{16n+5}=u_{8n+5}, && u_{16n+13}=u_{4n+1}\\
&u_{16n+6}=u_{8n+6}, && u_{16n+14}=u_{4n+2}\\
&u_{16n+7}=u_{4n+1}, && u_{16n+15}=u_{4n+3}
\end{aligned}
\tag{2}
$$

By projection the same relations hold for the sequence (v_n); hence (v_n) can be recursively defined by these relations (2) and the initial values $v_0, v_1,..., v_{15}$. It remains to check that

$$v_n=d'_{n+2} \qquad \text{for} \quad n=0, 1, 2,..., 15$$

and that

$$(v_n) \text{ satisfies } (2)$$

The first point is straightforward; the second point is longer to verify. To cut short a rather cumbersome proof, we shall only verify that the first relation in (2) holds for the sequence (d'_{n+2}): Let

$$F_n=d'_{n+2}=f'_{\varepsilon_{n+1}}\circ\cdots\circ f'_{\varepsilon_0}(d'_0)$$

for short

$$F_n = g_{n+1} g_n \cdots g_0(d'_0)$$

where $g_k = f'_{\varepsilon_k}$ and we omit the $\circ$'s.

Then, for n greater than or equal to 1,

$$F_{16n} = g_{16n+1}\, g_{16n}\, g_{16n-1}\, g_{16n-2} \cdots (d'_0)$$

$$F_{4n} = g_{4n+1}\, g_{4n}\, g_{4n-1}\, g_{4n-2} \cdots (d'_0)$$

The property of the Thue–Morse sequence implies that

$$g_{16n+1}\, g_{16n}\, g_{16n-1}\, g_{16n-2} = g_{4n+1}\, g_{4n}\, g_{4n-1}\, g_{4n-2} = h_n$$

Moreover one easily checks that for every choice of $\varepsilon_n = \pm 1$ and $\varepsilon_{n-1} = \pm 1$, the function h_n is a constant function.

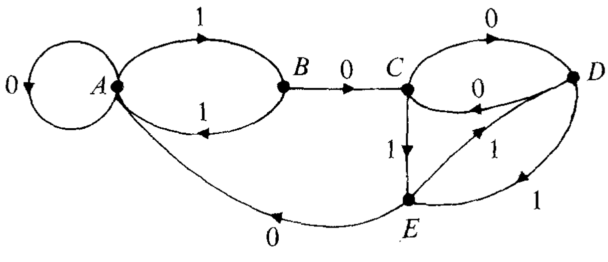
Finally, for every $n \geqslant 0$, $F_{16n} = F_{4n}$ (the case $n = 0$ is trivial).

After verifying the other 15 relations in (2) for F_n, we then conclude that

$$\forall n \geqslant 0, \qquad F_n = v_n$$

which was our claim.

We now conclude our paper with an automaton which generates the sequence (d'_{n+2}):



$T(A) = -$

$T(B) = T(D) = 0$

$T(C) = T(E) = +$

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